# Critical Invariant Galton-Watson Branching Process for Earthquake Occurrence

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## Summary

We propose a new theoretical modeling framework for seismicity based on a recently introduced family of invariant Galton-Watson (IGW) stochastic branching processes. The IGW framework overcomes some well-recognized problems in existing approaches while preserving the main postulates of stochastic branching models (Panel 1). Informally, the IGW model is the only family that is invariant with respect to multiple operations that represent imprecise observations and estimations of parameters in real data (Panel 2). The IWG framework provides a convenient approximation to modeling observed data with branching processes similar to the Epidemic Type Aftershock Sequence (ETAS) model. The theory of IGW processes suggests explicit distributions for multiple clustering statistics, including magnitude-dependent and magnitude-independent offspring number, cluster size, and cluster combinatorial depth. The framework provides a two-parameter model that can fit observed clusters of all sizes (not only the largest ones), and it suggests new observed statistics based on the Horton-Strahler analysis of earthquake clusters. The IGW model closely approximates the ETAS model, while allowing for a comprehensive theoretical analysis and more robust estimates of parameters. Analysis of seismicity in southern California (Panel 3) demonstrates that the IGW model provides a very close fit to observed earthquake clusters, and that the estimated IWG parameters and derived statistics are robust with respect to the lower magnitude threshold used in the analysis. The proposed model facilitates analyses of additional quantities of seismicity based on self-similar tree attributes, and may be used to assess the proximity of seismicity to criticality.

## 1. Background

### A. Branching Process Modeling of Earthquakes

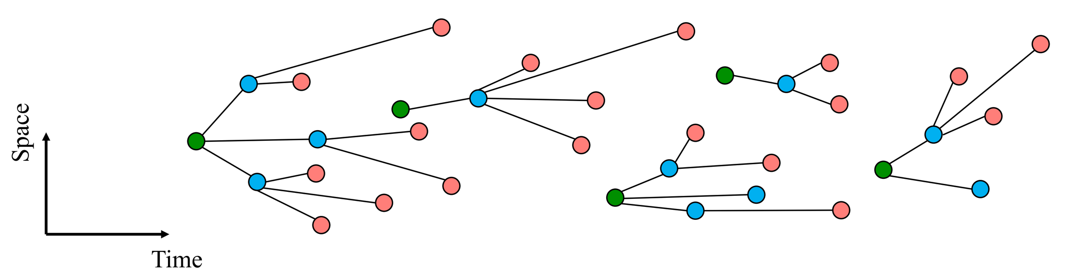
First models: Kagan (1973), Kagan & Knopoff (1976), Vere-Jones (1976)

Most widespread approach: Self-exciting Hawkes process that is equivalent to a branching process with immigration; Hawkes (1971), Hawkes and Oakes (1974), Adamopoulos (1976)

Space-time-magnitude generalization, statistical tools, applications to seismicity: Vere-Jones, Ogata, and collaborators (1970s-1980s)

Epidemic Type Aftershock Series (ETAS) model: Combines the key empirical laws of statistical seismology with rigorous modeling and inference tools Ogata (1988, 1989, 1999)

Idea of branching process models:




Immigrants (background) produce offspring (1st generation aftershocks) that produce their own offspring (2nd generation aftershocks), and so on…

Observations and statistical problem: The resulting process consists of background and all offspring. The event types need to be identified from observed clustering properties.

### B. Epidemic Type Aftershock Sequence Model (ETAS)

- Model describes a sequence of earthquakes with occurrence times $t_i$ and magnitudes $M_i \geq M_0$
- Background events form a Poisson process with intensity (rate) $\mu(t)$
- Magnitudes are assigned independently, according to the Gutenberg-Richter law

$$\mathsf{P}(M_i > M) = 10^{-b(M-M_0)}, \quad M > M_0$$

- Each earthquake with occurrence time $t_i$ and magnitude $M_i$ generates offspring according to a Poisson process with intensity (Omori-Utsu law)

$$\nu(t|t_i, M_i) = \frac{K_0 10^{\alpha(M_i - M_0)}}{(t - t_i + c)^p}, \quad t > t_i$$

- A combined earthquake flow includes the background events and their aftershocks of all generations – it is a point process with conditional intensity

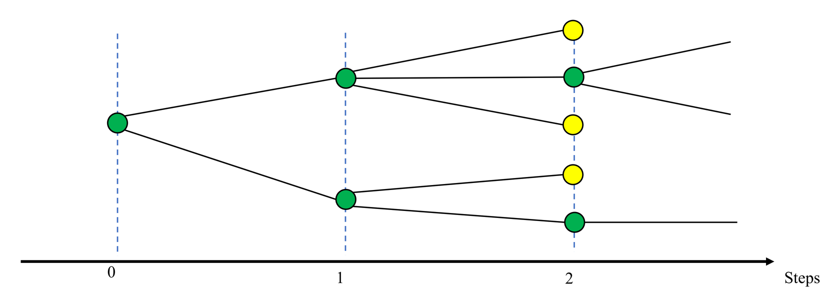
$$\lambda(t|\mathcal{H}_t) = \mu(t) + \sum_{i:t_i<t} \nu(t|t_i, M_i), \qquad \mathcal{H}_t = \{(t_i, M_i) : t_i < t\}$$

- A space component can be introduced (not discussed in this work)

## 2. Invariant Galton-Watson Process (IGW)

### A. Galton-Watson (GW) Process

- Describes a population that develops in discrete time; start with a single progenitor at step 0
- At every step, every member produces $k \geq 0$ offspring according to distribution $\{q_k\}$ and terminates.
- If the average progeny is unity ($\sum k q_k = 1$) the process is called *critical*
- If the average progeny is less than unity ($\sum k q_k < 1$) the process is called *subcritical*




### B. Invariant Galton-Watson Process (IGW): Definition

- IGW process is a special case of GW process specified by 2 parameters: $q$ and $r$
- IGW offspring number distribution

$q_1 = r$ (specifies probability of degree-2 chains)

$$q_k = (1-r)\frac{(1-q)\Gamma(k-1/q)}{q\Gamma(2-1/q)\,k!}, \quad k = 0, 2, 3, \ldots$$

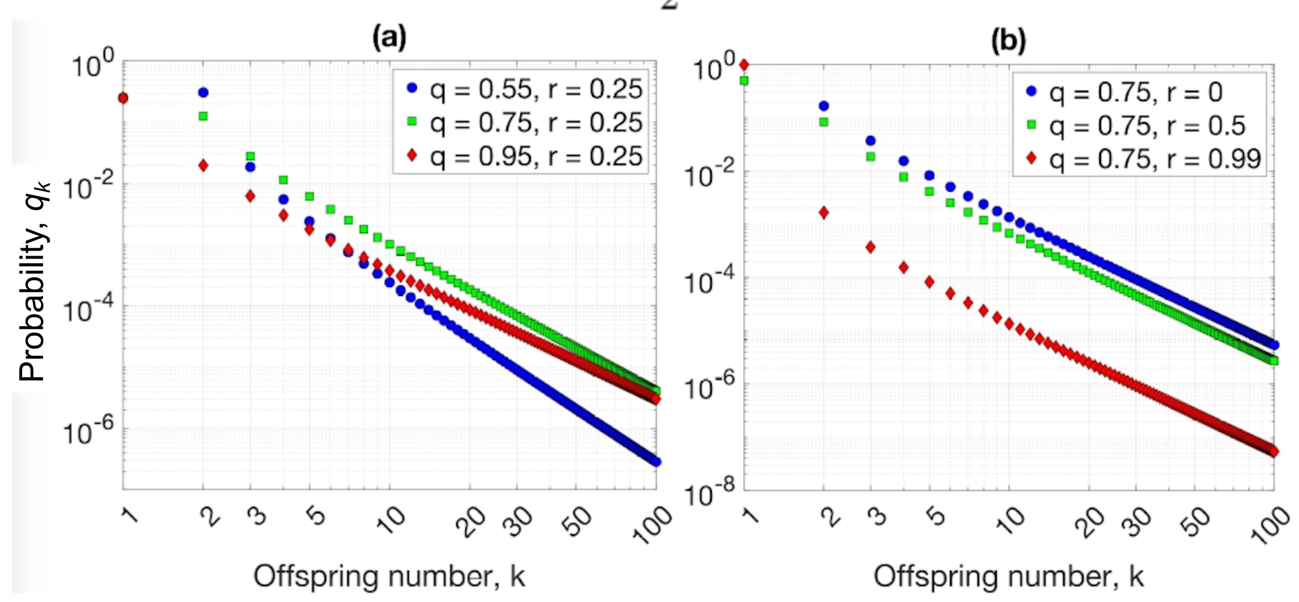
(specifies termination probability, $k$ = 0, and branching, $k$ > 1)

- This distribution has a Zipf-type power-law tail

$$q_k \sim Ck^{-(1+q)/q}, \quad C = (1-r)\frac{1-q}{q\,\Gamma(2-1/q)}.$$

- The IGW family includes the famous critical binary Galton-Watson process

$$\{q_0 = q_2 = \frac{1-r}{2}, \quad q_1 = r\}$$




**Examples of the IGW offspring distribution** with different parameters $q$ and $r$.
(a) Varying $q$ for a fixed $r$ = 0.25. (b) Varying $r$ for a fixed $q$ = 0.75.

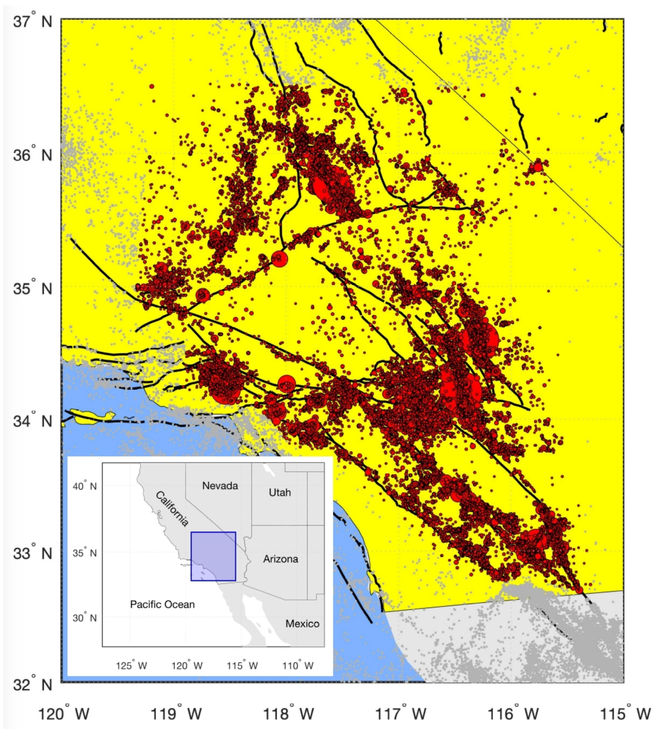
### C. Invariance property of IGW

- A trajectory of a Galton-Watson process is a tree $T$
- Consider a transformation that eliminates some subtrees from $T$. Often, a transformed tree is a trajectory of another Galton-Watson process; this is the case for the following transformations:
  - Continuous erasure from the leaves [Neveu,1986]
  - Minimal subtree that includes the root and a random set of leaves [Duquesne & Winkel, 2007]
  - Horton pruning (eliminates degree-2 chains connected to leaves) [Burd et al., 2000; Kovchegov and Zaliapin, 2020, 2021]
  - Generalized dynamical pruning [Kovchegov and Zaliapin, 2020; Kovchegov et al., 2021]
  - Hereditary reduction [Duquesne & Winkel, 2019]
- Such transformations represent imprecise observations and estimations of tree parameters in real data.
- IGW is the only family that is (a) invariant and (b) an attractor of critical Galton-Watson trees with respect to the above transformations

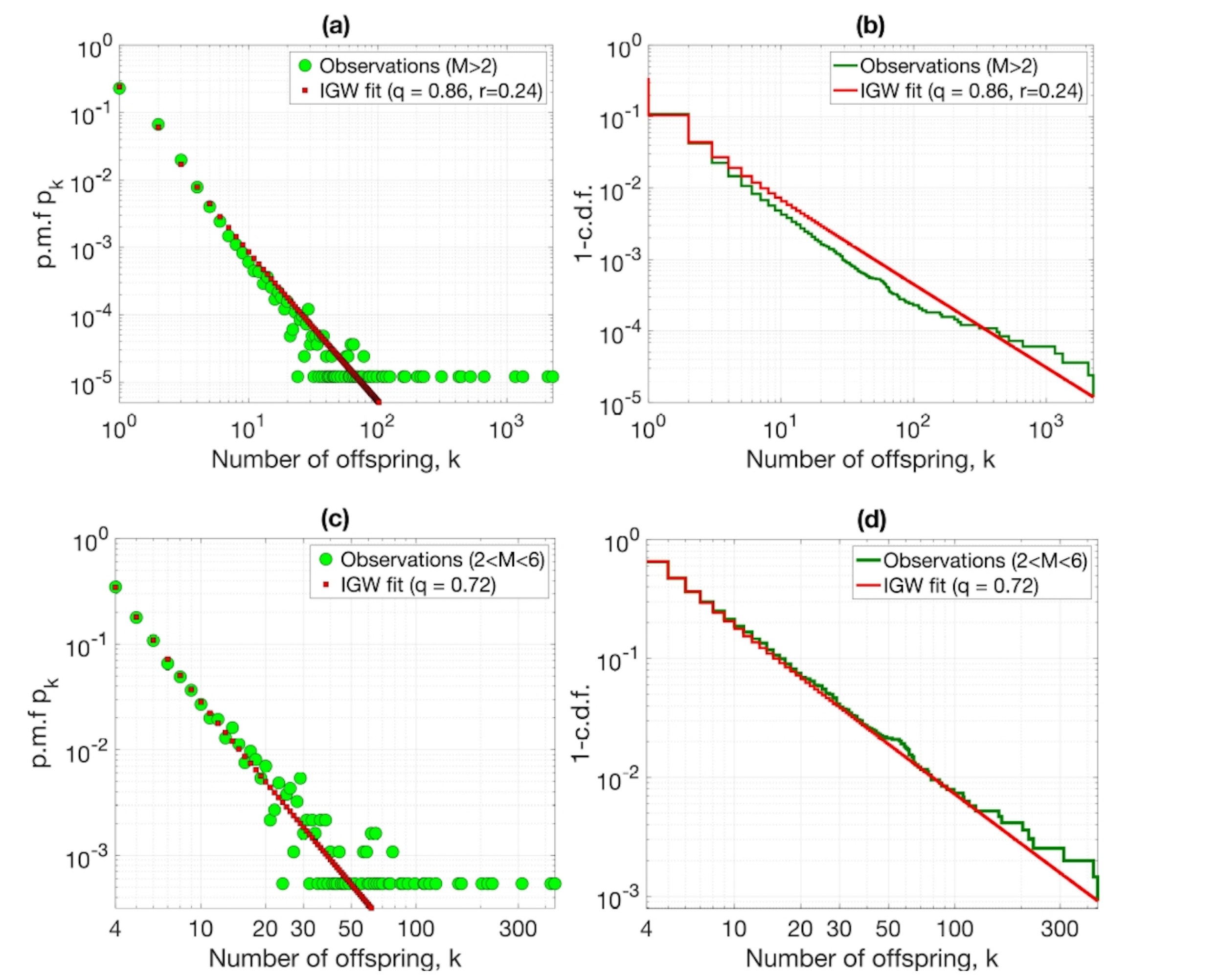
### D. IGW process: Analytical results

- IGW process approximates the ETAS (and similar branching processes with non-Poisson direct offspring distribution), with parameter $q = \alpha/b$ that combines the GR $b$-value and the aftershock productivity exponent $\alpha$ (see **Panel 1.B**)
- Many familiar statistics can be obtained analytically in the IGW process. This includes
  - Offspring numbers, cluster size, cluster combinatorial depth as well as multiple self-similar statistics
  - Horton-Strahler order of clusters, Horton law (numbers of branches of different orders), Side-branch numbers, Tokunaga coefficients
- This forms a flexible toolbox for tracking space-time changes in earthquake clustering

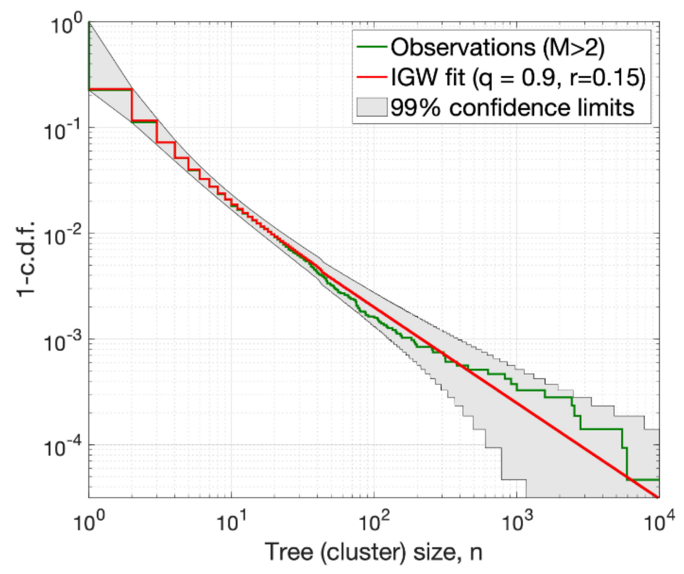
## 3. Analysis of seismicity in Southern California




**A. Seismicity of Southern California examined in this work**
Earthquakes with magnitude M > 2 in Hauksson et al. (2012) catalog extended to 1981-2019 are shown by gray dots, the examined earthquakes in the central part of the catalog are shown by red circles whose size is proportional to magnitude. Black lines show the major faults.

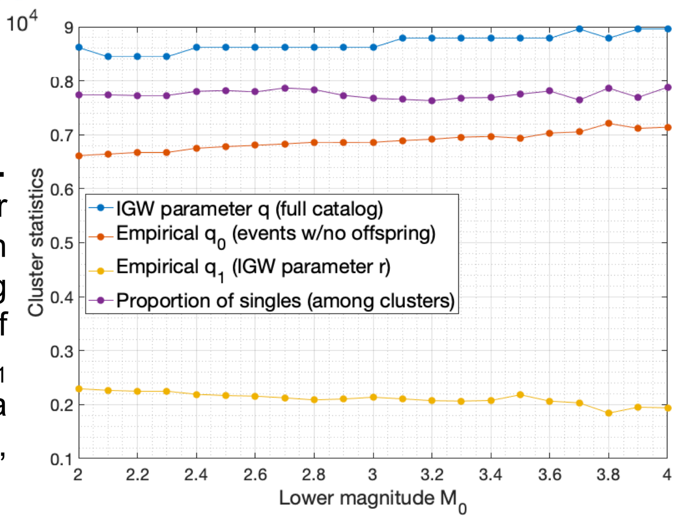



**B. IGW fit to the empirical offspring numbers (estimated)**
(a) p.m.f. for events with M > 2; the horizontal pattern of green circles in the bottom right corner corresponds to large offspring numbers that have been only observed once in the examined catalog
(b) survival function (1-c.d.f.) for M > 2
(c) p.m.f. conditioned on k > 3 for 2 < M < 6
(d) survival function (1-c.d.f.) conditioned on k > 3 for 2 < M < 6




**C. IGW fit to the empirical cluster sizes (number of tree vertices).**
The empirical survival function of cluster size (green) and its IGW fit (red). Grey color indicates 99% confidence region.

**D. Stability of empirical IGW statistics with respect to the cut-off magnitude $M_0$.**
The IGW parameter q estimated for empirical off spring numbers (green circles), proportion of singles among clusters (blue squares), proportion $q_0$ of events with no offspring, and proportion $q_1$ = r of events with a single offspring as a function of the cutoff magnitude $M_0$ = 2.0, 2.1, . . . , 3.9, 4.0.

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